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TIME-INTEGRATION SCHEMES FOR METEOROLOGICAL SIMULATIONS INVOLVING OSCILLATORY AND DISSIPATIVE MODES

by
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July 1990

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ABSTRACT

Time integration schemes for numerical weather prediction are sought that each have the following properties: stability over the range of complex frequencies characteristic of both dissipative and stable oscillatory modes; a requirement for only a single evaluation of the nonlinear forcing per time step; small truncation errors for slow modes.

From a preliminary examination of a small selection of explicit methods, one, a hybrid comprising a leapfrog predictor and trapezoidal corrector, appears especially promising for the integration of the atmospheric (or oceanic) primitive equations, being accurate and stable for slow, neutrally stable or slightly dissipative, quasi-geostrophic motions, while generalising very naturally to a semi-implicit method to accommodate modes whose oscillatory or dissipative time scales are significantly shorter than the time scale appropriate for accurate representation of the quasi-geostrophic evolution. A version of this generalisation is shown to be robust to minor mis-specifications of the projection of the system-Jacobian assumed for those components treated implicitly.

1. INTRODUCTION

A system as complicated as a numerical weather prediction (NWP) model, linearised about a state of well developed realistically balanced motion, includes among its modes many of a chiefly oscillatory character, some that are to varying degrees dissipative, and a few that are mildly unstable. Numerical integration schemes able to do justice to the variety of modes present in such a system must either comprise a number of effectively distinct and specialised parts, as is commonly the case for many operational forecasting models, or else they must be robust to the broad range of characteristic frequencies of the linearised excitations that realistic motions of the system can engender.

The present note initially examines simple schemes that attempt to fulfil the latter criterion, that is, stability to a set of intrinsic frequencies (which are in general complex). An additional criterion greatly reducing the number of schemes considered here is that each one should involve only a single nonlinear function evaluation for the forcing of the system per timestep.

The motivation for focusing attention on schemes involving a single new evaluation of the forcing per time step stems from the recognition that, in present day models that incorporate a thorough parametrisation of many complicated physical processes and whose dynamics may be represented using relatively costly high-order spatial differencing, the attainment of numerical efficiency tends to preclude iterative evaluation of the forcings unless compelling extenuating reasons supervene.

Since it is assumed that the typical evolution of the system is too highly nonlinear to make possible the exact inversion of the ever-changing Jacobian associated with a fully implicit scheme, a further criterion of the basic schemes tested in section 2 is that all should be explicit methods. While this study makes no pretence to be as comprehensive as the previous investigations of time integration techniques due to Kurihara (1965) and Young (1968), the few candidate schemes that satisfy the restrictive terms of reference of my investigation include simple, yet viable, methods not discussed by the aforementioned authors.

All the schemes defined in section 2 are isomorphic to linear "multistep" methods (for a general introduction of these and other methods in the context of ordinary differential equations see Gear, 1971). Two of these are the Adams-Bashforth methods of orders two and three. One subset of the schemes described consists of the leapfrog method augmented (as is the usual practice) with the Robert-Asselin time-filter (Robert, 1966; Asselin, 1972). Each value of the filter parameter σ implies a particular integration scheme with its own unique stability and accuracy characteristics. A larger parameter σ yields a scheme more robust to the presence of dissipation, but at the expense of a larger coefficient of the assymptotic first order truncation error. One method that I shall refer to as the "leapfrog-trapezoidal" scheme (although it is slightly different from the scheme of the same name included in the study of Kurihara), does not correspond exactly to any standard multistep formula, but it is a close relative of the third-order Adams-Bashforth method. Like the third-order Adams-Bashforth method, it requires storage sufficient for four fields of data (compared to three required for the time-filtered leapfrog), yet it

can boast only second-order accuracy and has a marginally more restrictive stability domain than the Adams-Bashforth method. Despite these apparently disqualifying attributes, the algorithmic structure of the leapfrog-trapezoidal method naturally lends itself, in the context of a primitive equations model, to a generalisation in the form of an efficient, stable, and uniformly accurate, semi-implicit scheme that, in contrast to any corresponding semi-implicit generalisation of the Adams-Bashforth method, requires practically no additional storage. The robustness of this implicit leapfrog-trapezoidal scheme is explored in section 3 and potential applications are discussed in section 4.

2. ANALYSIS OF SIMPLE EXPLICIT INTEGRATION SCHEMES REQUIRING A SINGLE FUNCTION EVALUATION PER TIME STEP

A prototype for linearised unforced dynamical systems is the one-mode sytem,

$$\frac{d\psi}{dt} = \nu\psi \quad , \tag{2.1}$$

with "frequency" ν a complex constant. Much of the sensitivity of a more general dynamical system, for example, one governed by a coupled vector equation,

$$\frac{d\underset{\sim}{\psi}}{dt} = \underset{\sim}{F}(\underset{\sim}{\psi}, t) \quad , \tag{2.2}$$

where F is a nonlinear function of both ψ and time t, can be adequately simulated for the purpose of testing numerical schemes, by appeal to a number of simple systems of the form (2.1) with appropriate values for ν. Further generalisation is achieved by augmenting (2.1) with a periodic forcing, or by prescribing a time variation of ν, but such extensions will not be used here.

For systems describing large-scale meteorological motion, in which quasi-uniform advection plays a prominent role, the appropriate idealisation (2.1) of modes of the system involves frequencies ν that are purely imaginary. Other terms, especially those associated with dissipative boundary layer processes, are more appropriately modelled with negative real values, ν. In any case, when the dynamical or physical processes themselves are intrinsically stable (equivalent to the real part of the appropriate ν being non-positive), it is desirable that the numerical representation of them by the time-integration scheme is correspondingly stable. Pertinent to questions of stability and accuracy are the results for multistep methods summarised in the classic study of Dahlquist (1963): no explicit method, nor any implicit multistep method of order greater than two, is stable for all stable ν; of the second-order accurate schemes that are stable for all stable ν, the one with smallest coefficient of asymptotic truncation error is the implicit trapezoidal scheme. For NWP, a second order time integration scheme (such as the unfiltered leapfrog or implicit trapezoidal) is invariably considered sufficiently accurate, and even a first-order method (such as the time-filtered leapfrog) might be acceptable provided the principal coefficient of error is sufficiently small. While not expecting stability at large absolute values of ν, a

viable scheme for a system that possesses both neutral and damped waves must at least be stable for a region of complex ν that includes a proper rectangle bounded to the right by the imaginary axis, that is, a set:

$$\{ \nu : \mathrm{Re}(\nu) \in [-a , 0] , \mathrm{Im}(\nu) \in [-b , b] \} \quad , \tag{2.3}$$

for some positive a and b.

Below, a number of schemes are introduced, the choice restricted by the criteria listed in the introduction. In each case the state of the dynamical system at a time, $t = \tau \, \delta t$, after the start of the integration, is denoted by ψ^{τ}, where δt is the time step. The forcing corresponding to this state is denoted F^{τ}. ψ^{*} refers to a preliminary estimate of the quantity, ψ, with F^{*} computed from to ψ^{*}. In general, both ψ and F are multi-component vectors, the Jacobian of F with respect to ψ being some time-dependent matrix,

$$\nu = \frac{\partial(F)}{\partial(\psi)} \quad . \tag{2.4}$$

But, in the context of each idealised single mode homogeneous system (2.1) ν reduces to a constant complex scalar [an eigenvalue of the typical instantaneous Jacobian of (2.4)].

a. Definitions of basic schemes

1). Scheme 1

$$\psi^{*\tau+1} = \psi^{\tau} + F^{*\tau}\,\delta t \ , \tag{2.5a}$$

$$\psi^{\tau+1} = \psi^{\tau} + F^{*\tau+1}\,\delta t \ , \tag{2.5b}$$

This scheme superficially resembles the Euler-backward method (Kurihara, 1965). However, the function evaluation to obtain F^* is not iterated in the present method.

2). Scheme 2

Replacing the corrector of (2.5) with the more accurate trapezoidal form, one obtains the "forward-trapezoidal" scheme (like the "Heun" scheme, but without the iterated function evaluation):

$$\psi^{*\tau+1} = \psi^{\tau} + F^{*\tau}\,\delta t \ , \tag{2.6a}$$

$$\psi^{\tau+1} = \psi^{\tau} + \tfrac{1}{2}(F^{*\tau+1} + F^{*\tau})\,\delta t \ . \tag{2.6b}$$

In the appendix this method is shown to be isomorphic to the second-order Adams-Bashforth method.

3). Scheme 3

The third-order Adams-Bashforth scheme (for example, Gear, 1971) is:

$$\psi^{\tau+3} = \psi^{\tau+2} + (23F^{\tau+2} - 16F^{\tau+1} + 5F^{\tau})\,\delta t/12 \ . \tag{2.7}$$

4). Scheme 4

Replacing the first-order forward predictor formula of (2.6) by a second-order leapfrog step might be expected to improve accuracy, if not stability. The resulting leapfrog-trapezoidal scheme is:

$$\psi^{*\tau+2} = \psi^{\tau} + 2F^{*\tau+1}\,\delta t\ , \tag{2.8a}$$

$$\psi^{\tau+2} = \psi^{\tau+1} + \tfrac{1}{2}(F^{*\tau+1} + F^{*\tau+2})\,\delta t\ . \tag{2.8b}$$

Again, the fact that the nonlinear forcing F remains uniterated means this scheme is not quite identical to the superficially similar leapfrog-trapezoidal scheme discussed by Kurihara. At first sight this scheme bears no obvious resemblance to the third-order Adams-Bashforth method (2.7) but in the appendix it is shown that, considering ψ^* to be the final variable instead of ψ, an equivalent representation of (2.8) takes the same basic form as (2.7) except with coefficients that are numerically slightly different.

5). Scheme 5

$$\psi^{*\tau+2} = \psi^{\tau} + 2F^{*\tau+1}\,\delta t\ , \tag{2.9a}$$

$$\psi^{\tau+1} = \psi^{*\tau+1} + \tfrac{1}{2}\sigma(\psi^{*\tau+2} - 2\psi^{*\tau+1} + \psi^{\tau})\ . \tag{2.9b}$$

The corrector formula, (2.9b), may be expressed equivalently,

$$\psi^{\tau+1} = \psi^{*\tau+1} + \sigma(\psi^{*\tau+2} - \psi^{*\tau+1} - F^{*\tau+1}\,\delta t)\ . \tag{2.9c}$$

This scheme is familiar as the leapfrog, generalised by the application of the Robert-Asselin time-filter with parameter σ (Robert, 1966; Asselin, 1972). For the special choice, $\sigma = 0$, it reduces to the pure leapfrog method.

b. Accuracy

In each case assume the forcing F^τ takes the form,

$$F^\tau = \nu \psi^\tau \ . \qquad (2.10)$$

Assume each preliminary estimate $\psi^{*\tau}$ to be related to the final estimate ψ^τ by a factor, a, and let $F^{*\tau}$ be related to ψ^τ by a factor ν^*:

$$\psi^{*\tau} = a \psi^\tau \ ,$$
$$F^{*\tau} = \nu^* \psi = \nu \psi^* \ . \qquad (2.11)$$

Also, for each homogeneous mode of the numerical representation, let the factor by which the solution changes over one time step be written simply as ψ, that is,

$$\psi = \psi^{\tau+1}/\psi^\tau \qquad (2.12)$$

Solutions for which, $|\psi| > 1$, are numerically unstable. To depict the region in the complex ν-plane for which each scheme is stable it is therefore sufficient to plot the locus of ν as ψ traces out the unit circle, centred at the origin, in the complex ψ-plane, that is,

$$\psi = \exp(\nu') \tag{2.13}$$

with ν' a pure imaginary between $\pm\pi i$.

In order to simplify the analysis of the remainder of this section the time step δt is taken to be unity.

1) Scheme 1

From (2.5):

$$a\psi = 1 + \nu^* \quad , \qquad \psi = 1 + \nu^*\psi \quad . \tag{2.14}$$

Thus,

$$a = \frac{2\psi - 1}{\psi^2} \equiv 1 - \nu'^2 + O(\nu'^3) \tag{2.15}$$

$$\nu = \frac{\nu'}{a} \equiv \frac{\psi(\psi-1)}{2\psi-1} \equiv \nu'[1 - \tfrac{1}{2}\nu' + O(\nu'^2)] \tag{2.16}$$

So, scheme 1 is only first-order accurate and the leading coefficient of error is not particularly small. On the basis of its poor accuracy, this scheme would not be considered for applications in NWP.

2). Scheme 2

$$a\psi = 1 + v' \quad ,$$

$$\psi = 1 + \tfrac{1}{2}v'^{*}\psi + \tfrac{1}{2}v'^{*} \quad , \tag{2.17}$$

implying,

$$a = \frac{3\psi - 1}{\psi(\psi+1)} \equiv 1 - \tfrac{1}{2}v'^{2} + O(v'^{3}) \quad , \tag{2.18}$$

$$v = \frac{v'^{*}}{a} \equiv \frac{2\psi(\psi-1)}{3\psi-1} \equiv v'\left[1 + \frac{5v'^{2}}{12} + O(v'^{3})\right] \quad . \tag{2.19}$$

The accuracy of the method for free modes in the vicinity of $v = 0$ is second-order with coefficient of error (=5/12) five times larger in absolute magnitude that of the corresponding coefficient of error (= -1/12) for the implicit trapezoidal method. For forced motion at limitingly small frequencies the final estimate ψ is likewise five times as accurate as the preliminary estimate ψ^{*}.

3). Scheme 3

$$\psi^{3} = \psi^{2} + \frac{v}{12}(23\psi^{2} - 16\psi + 5\psi) \quad , \tag{2.20}$$

that is,

$$v = \frac{12\psi^{2}(\psi - 1)}{23\psi^{2} - 16\psi + 5} = v'\left[1 + \frac{3}{8}v'^{3} + O(v'^{4})\right] \quad . \tag{2.21}$$

Thus, the coefficient of third-order error for this Adams-Bashforth method is 3/8.

4. Scheme 4

$$a\psi^2 = 1 + 2\,\nu^*\psi \quad , \qquad (2.22)$$

$$\psi = 1 + \tfrac{1}{2}\,\nu^*(\psi + 1) \quad .$$

So,

$$a = \frac{4\psi^2 - 3\psi + 1}{\psi^2(\psi + 1)} \equiv 1 - \tfrac{1}{2}\nu'^2 + O(\nu'^4) \quad , \qquad (2.23)$$

$$\nu = \frac{\nu^*}{a} = \frac{2\psi^2(\psi - 1)}{4\psi^2 - 3\psi + 1} \qquad \nu'\left[\,1 - \frac{1}{12}\nu'^2 + O(\nu'^3)\,\right] \quad . \qquad (2.24)$$

The scheme is therefore second-order accurate with error coefficient (= -1/12) equal to that of the trapezoidal method. Since the predictor and corrector solutions differ only to third order in ν', both estimates are essentially of equivalent accuracy. Note the similarity in the form of the expressions for ν in terms of ψ for (2.21) and (2.24).

5. Scheme 5

$$a\psi^2 = 1 + 2\,\nu^*\,\psi \quad , \qquad (2.25)$$

$$1 = a\,[\,1 + \sigma(\psi - 1)\,] - \sigma\nu^* \quad .$$

Hence,

$$a = \frac{2\psi - \sigma}{\psi(\psi\sigma + 2 - 2\sigma)} = 1 - \frac{(1 - \sigma)\nu' + O(\nu'^2)}{(2 - \sigma)} \quad , \qquad (2.26)$$

$$\nu = \frac{\nu^*}{a} \equiv \frac{(\psi - 1)(\psi + 1 - \sigma)}{2\psi - \sigma} = \nu'\left[\,1 - \frac{\sigma}{2(2-\sigma)}\nu' + O(\nu'^2)\,\right] \quad , \qquad (2.27)$$

which implies a first-order scheme for $\sigma > 0$, with coefficient of error roughly proportional to σ for the typical small values of this parameter. In the case of the pure leapfrog scheme, $\sigma = 0$, then,

$$\nu = \nu'\left[\,1 + \frac{1}{6}\nu'^2 + O(\nu'^3)\,\right] \quad , \qquad (2.28)$$

which therefore implies a scheme accurate to second-order but with truncation error twice as large (and opposite in sign) to that of the trapezoidal method and scheme 4.

c. Stability

Using (2.16), (2.19), (2.21) and (2.24) with ψ allowed to trace out all complex values of unit magnitude, the loci of ν are found corresponding to neutrally stable cases of the respective schemes 1, 2, 3 and 4. These curves in the complex ν-plane are plotted in figure 1. Only the upper half of the full diagram is shown, the lower half being merely a mirror reflection of it. Scheme 1, although already disqualified because of its poor truncation error, does at least satisfy the stability criterion (2.3) with b as large as $1/\sqrt{3}$. Scheme 2 (equivalent to the second-order Adams-Bashforth method), although more accurate and with a larger overall region of stability, does not satisfy the requirements of stability for pure-oscillatory modes and thus, it is not a practical contender for applications in NWP. Scheme 3 is the third-order Adams-Bashforth method, and enjoys both excellent accuracy, as was shown in the previous subsection, together with the best stability characteristics of these first four schemes. For oscillatory modes the stability limits are $\nu = \pm i(b_{max})$ where b_{max} exceeds 0.7 . The leapfrog-trapezoidal method, scheme 4, has a similar, though marginally smaller, stability region, the limits on the imaginary axis in this case being given by $\nu = \pm i2/3$.

For the pure leapfrog method, i.e., with $\sigma = 0$, the stability region collapses to a segment on the imaginary axis between $\nu = \pm i$, the

computational mode being the unstable one for values of v to the left of this segment. The unfiltered leapfrog scheme clearly has no robustness to the presence of damping and must be combined with some method to control the computational mode. Because of the free parameter for the filtered leapfrog scheme, the stability limits for v have been traced out for three separate choices: a) $\sigma = 0.1$; b) $\sigma = 0.2$; c) $\sigma = 0.5$. These curves are indicated in figure 2. It is evident from these curves that, in order to incorporate enough time-filtering to make the scheme robust to relatively significant amounts of dissipation, a correspondingly large filter parameter σ must be used. The penalty paid for this is a concomitantly large first-order truncation error.

d. Discussion

Among the simple explicit integration schemes considered, the filtered leapfrog, the third-order Adams-Bashforth and the leapfrog-trapezoidal all merit use in systems, such as NWP models, which contains both oscillatory and dissipative modes. Of these, the filtered leapfrog method requires storage equivalent to three fields of data at one time while the two other contenders require storage for four fields. Thus, the gains in formal accuracy and stability acquired by using the third-order Adams-Bashforth or leapfrog-trapezoidal schemes are paid for by an increase in the space needed. From the results obtained so far the leapfrog-trapezoidal scheme is inferior in every respect when measured against the Adams-Bashforth scheme; it would hardly merit further attention were it not for the fortuitous fact that its algorithmic structure leads to a useful semi-implicit generalisation applicable to the case of governing equations which support latent or actual fast modes which, without an implicit treatment, would

force the adoption of unduly small timesteps. Moreover, the implementation of the semi-implicit leapfrog-trapezoidal scheme can be achieved with essentially no additional storage penalty and in this respect it cannot be conveniently matched by any corresponding implicit generalisation of the Adams-Bashforth technique. The following section pays particular attention to the stability of the semi-implicit leapfrog-trapezoidal method.

3. SEMI-IMPLICIT LEAPFROG-TRAPEZOIDAL METHOD

The implicit trapezoidal method is shown by Dahlquist (1963) to be the most accurate linear multistep method stable for all frequencies ν lying in the half plane: $Re(\nu) \leq 0$. The corrector formula (2.8b) is sufficiently similar to the true implicit trapezoidal method that it requires only a minor modification to form the semi-implicit procedure. The predictor, (2.8a), then formally becomes redundant for those modes treated implicitly, provided ν_o exactly equals ν. In practice only very few deepest vertical gravity modes of an NWP model need the benefits of an implicit treatment (Burridge, 1975) and it is easier to accommodate the relevant adjustments after the existing corrector step (2.8b) has been executed for <u>all</u> modes. The steps of such an algorithm are set out as follows:

$$[A] \quad \psi^{*\tau+2} = \psi^{\tau} + 2F^{\dagger\tau+1}\delta t \quad , \tag{3.1a}$$

$$[B] \quad F^{*\tau+2} = F(\psi^{*\tau+2}) \quad , \tag{3.1b}$$

$$[D] \quad R^{\tau+2} = \psi^{*\tau+2} - \psi^{\tau+1} - \tfrac{1}{2}(F^{\dagger\tau+1} + F^{*\tau+2}) \quad , \tag{3.1c}$$

$$[A] \quad \psi^{**\tau+2} = \psi^{*\tau+2} - R^{\tau+2} \quad , \tag{3.1d}$$

$$[D]\ \Delta F^{\tau+2} = -\ v_o\ (\ I - \tfrac{1}{2} v_o\ \delta t\ )^{-1}\ \ R^{\tau+2}\ , \qquad (3.1e)$$

$$[A]\ \psi^{\tau+2} = \psi^{*\tau+2} + \tfrac{1}{2}\Delta F^{\tau+2}\ \delta t\ , \qquad (3.1f)$$

$$[B]\ \ F^{\tau+2} = F^{*\tau+2} + \Delta F^{\tau+2}\ , \qquad (3.1g)$$

$$[C]\ \psi^{*\tau+3} = \psi^{\tau+1} + 2F^{\tau+2}\delta t\ . \qquad (3.1a')$$

The bracketed labels, [A], [B], [C], [D], preceeding each of (3.1) signify one of the four blocks of storage set aside to hold the new field computed in that equation. In this case, it is assumed that, prior to evaluation of (3.1a), [A] contains ψ^{τ}, [C] contains $\psi^{\tau+1}$, [D] contains $F^{\tau+1}$. The last evaluation, (3.1a'), merely begins the cycle for the next time step, but with [C] and [D] substituted for [A] and [B] and vice-versa. Operator v_o appearing in (3.1e) denotes the approximated projection of Jacobian (2.4) into the space of those fast modes receiving the implicit treatment - for all remaining modes this operator effectively vanishes making (3.1e), (3.1f) and (3.1g) superfluous. In the context of the hydrostatic primitive equations with the linearisation to yield v_o being about a balanced state of rest, (3.1e) involves the inversion of one or more elliptic equations for a grid point model, as in other semi-implicit procedures (eg., Kwizak and Robert, 1971). With the availability of fast (spectral or multigrid) elliptic solvers, this stage need not be inordinately expensive. The main burden of computation, as in the explicit schemes, continues to fall upon the single nonlinear step, (3.1b), representing the function evaluation.

While the stability of the numerical procedure (3.1) is guaranteed when the presumed Jacobian v_o exactly equals the actual one, v, for all stable v (i.e., for all its eigenvalues on or to the left of the imaginary axis), the practical stability question concerns the case when v_o is an imperfect approximation. For example, the frequencies of gravity waves can be raised or lowered by Doppler shifting due to the ambient flow but it is practically impossible to vary v_o directly to allow for this. It is therefore appropriate to consider the idealised system of a single mode implied by (3.1) when v_o is fixed but v remains free to vary. The analysis proceeds as in section 2 and, as before, it will be convenient to assume $\delta t = 1$. Let the relationship of R^τ and $F^{+\tau}$ to ψ^τ for the free homogeneous numerical modes be,

$$R^\tau = r\psi^\tau \quad , \qquad F^{+\tau} = v^+\psi^\tau \quad . \tag{3.2}$$

Then proceding as in subsection 2(c), the steps of (3.1) imply:

$$a\psi^2 = 1 + 2\, v^+\psi \quad ,$$

$$r\psi = a\psi - 1 - \tfrac{1}{2}(v^+ + v^{*}\psi) \quad ,$$

$$1 = a - \frac{2r}{2-v_o} \quad , \qquad v^+ = v^{*} - \frac{2v_o r}{2- v_o} \quad , \tag{3.3}$$

from which is obtained (2.23) as before together with a new formula

relating ν to ψ:

$$\nu = \frac{\nu^*}{a} = \frac{2\psi^2(\psi - 1) - \nu_o^2(\psi - 1)^2}{4\psi^2 - 3\psi + 1} \qquad (3.4)$$

The scheme trivially retains second-order accuracy for asymptotically small ν'. The stability for ν in the neighbourhood of ν_o is more interesting, particularly for the neutrally stable oscillatory modes (ν_o on the imaginary axis). Figures 3(a), (b) and (c) show the stability curves in the ν-plane for $\nu_o = i$, $2i$ and $3i$ respectively. In each case only the interior and boundary of the upper lobe corresponds to values ν that yield a numerically stable scheme. It is observed that any neighbourhood of ν_o on the imaginary axis contains ν for which the scheme is slightly unstable; for small values ν_o the instability may be imperceptible but it cannot be ignored for larger ν_o that also differ significantly from the true system frequency, ν. The stability range on the imaginary axis in these examples has limits at $\nu = \nu_o$ and $\nu = 2(\nu_o + i)/3$. By augmenting ν_o with a small positive real part, stability is recovered on both sides of the intended frequency. This is seen in each of figures 4(a), (b) and (c), which correspond to figures 3(a), (b) and (c) respectively except for the addition of an increment of 0.1 to each of the corresponding values of ν_o.

4. SUMMARY

Among some efficient and simple time integration schemes one, a "leapfrog-trapezoidal" method, stands out as being attractive for accurate numerical simulation of atmospheric or oceanic systems that contain both oscillatory and damped modes, being stable without modification, to these

modes when their time scales are well resolved by the time step chosen, while permitting the necessary semi-implicit modification to be made in a particularly straightforward way when some of their time scales are short enough to require an implicit treatment. While it is recognized that the cost of reserving four times the storage needed for a single model field remains too high a price for many existing operational or experimental models whose success or competiveness relies on achieving the maximum spatial resolution possible, the increasing availability of computers with large amounts of central memory together with a perceptible trend towards parallel integrations of ensembles of NWP models to accommodate the needs of medium-range prediction, means that in future models, a greater stress may be placed on efficiency and accuracy of the computations rather than on the attainment of maximum achievable resolution.

Historically, truncation errors in primitive equations models have been mainly a problem involving the spatial differencing; temporal differencing having been immune by virtue of the excessively small time steps employed for stabilty (not accuracy) reasons. Improvements in spatial accuracy brought about by developments of spectral and finite-element techniques, or by the adoption of high-order differencing, together with the recent resurgence of interest in semi-Lagrangian methods that open the way to the use of longer time steps, all serve to highlight the need to ensure that time discretisations, just as much as their spatial counterparts, preserve comparably high standards of numerical accuracy and stability. In this note I have drawn attention to the merits of one particular time integration scheme which is simple to apply, frugal on function evaluations, virtually as accurate as the pure trapezoidal method,

and is, or at least can be easily adapted to be, stable to an extensive range of both oscillatory and dissipative modes.

APPENDIX

Multistep methods equivalent to the first four schemes of section 2.

Using (2.5a) to eliminate ψ^{n} and ψ^{n+1} from (2.5b) yields a single explicit multistep formula for the preliminary value:

$$\psi^{*n+2} = \psi^{*n+1} + 2F^{*n+1} - F^{*n} \quad . \tag{A.1}$$

The same substitution into (2.6b) yields

$$\psi^{*n+2} = \psi^{*n+1} + \frac{3}{2}F^{*n+1} - \frac{1}{2}F^{n} \quad . \tag{A.2}$$

which is just the second-order Adams-Bashforth method. Using (2.8a) to eliminate ψ^{*n+1} and ψ^{*n+2} from (2.8b) leads to:

$$\psi^{n+2} = \psi^{*n+2} + \tfrac{1}{2}(4F^{*n+2} - 3F^{*n+1} + F^{*n}) \quad , \tag{A.3}$$

which approximates the form of the Adams-Bashforth method (2.7).

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FIGURE CAPTIONS

Figure 1. Boundaries of regions in the complex v-plane within which numerical integration schemes 1, 2, 3 and 4, as indicated, remain stable.

Figure 2. As for figure 1 but for the time-filtered leapfrog schemes. Curve a corresponds to $\sigma = 0.1$, curve b to $\sigma = 0.2$ and curve c to $\sigma = 0.5$.

Figure 3. The stability regions corresponding to the algorithm of (3.1) are within the upper loops displayed for the following values of v_o: (a) $v_o = i$; (b) $v_o = 2i$; (c) $v_o = 3i$.

Figure 4. As for figure 3 but for v_o possessing a small positive real component: (a) $v_o = 0.1 + i$; (b) $v_o = 0.1 + 2i$; (c) $v_o = 0.1 + 3i$.

Figure 1

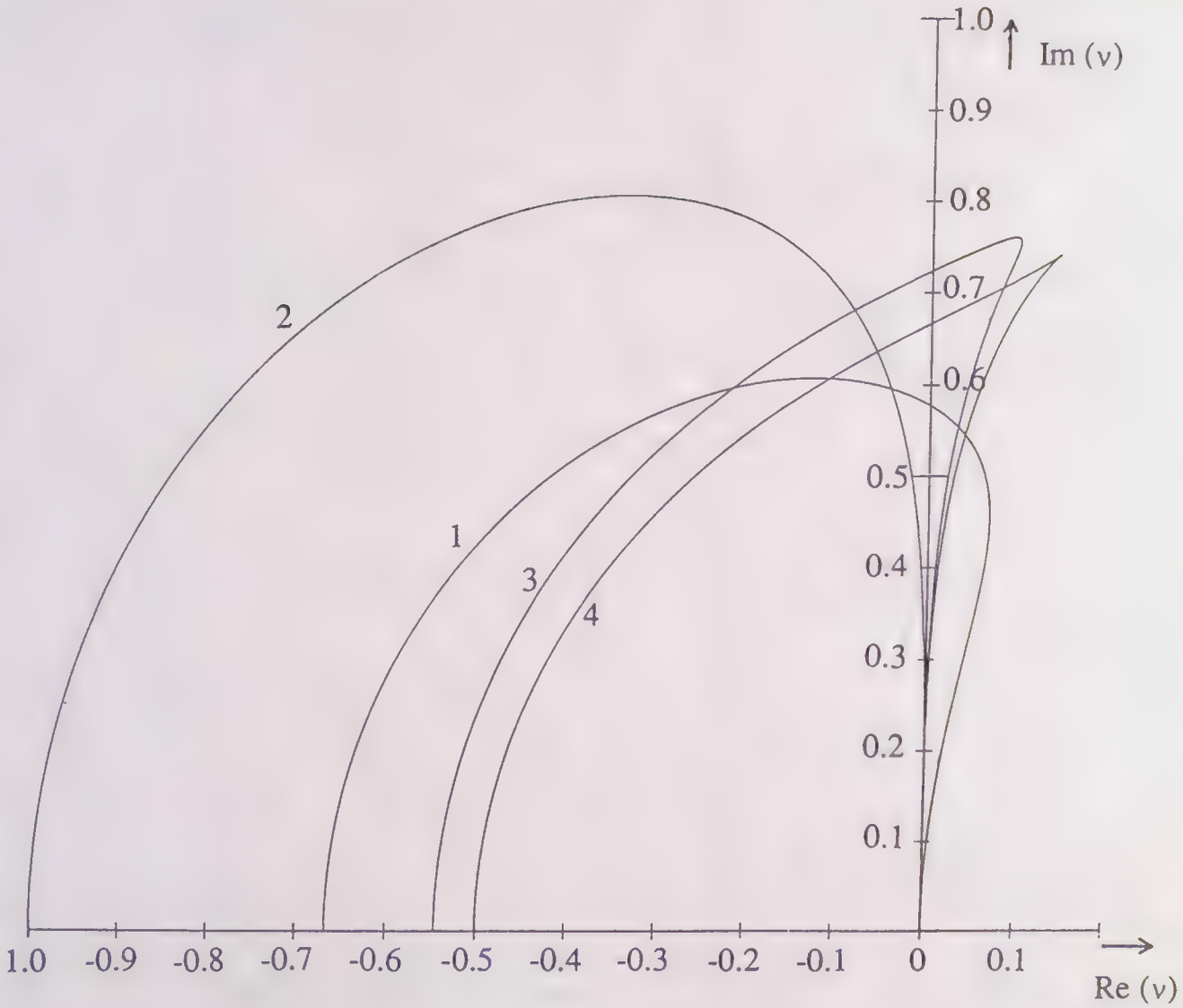

1.0
Im (ν)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
1
3
4
1.0 -0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0 0.1
Re (ν)

Figure 2

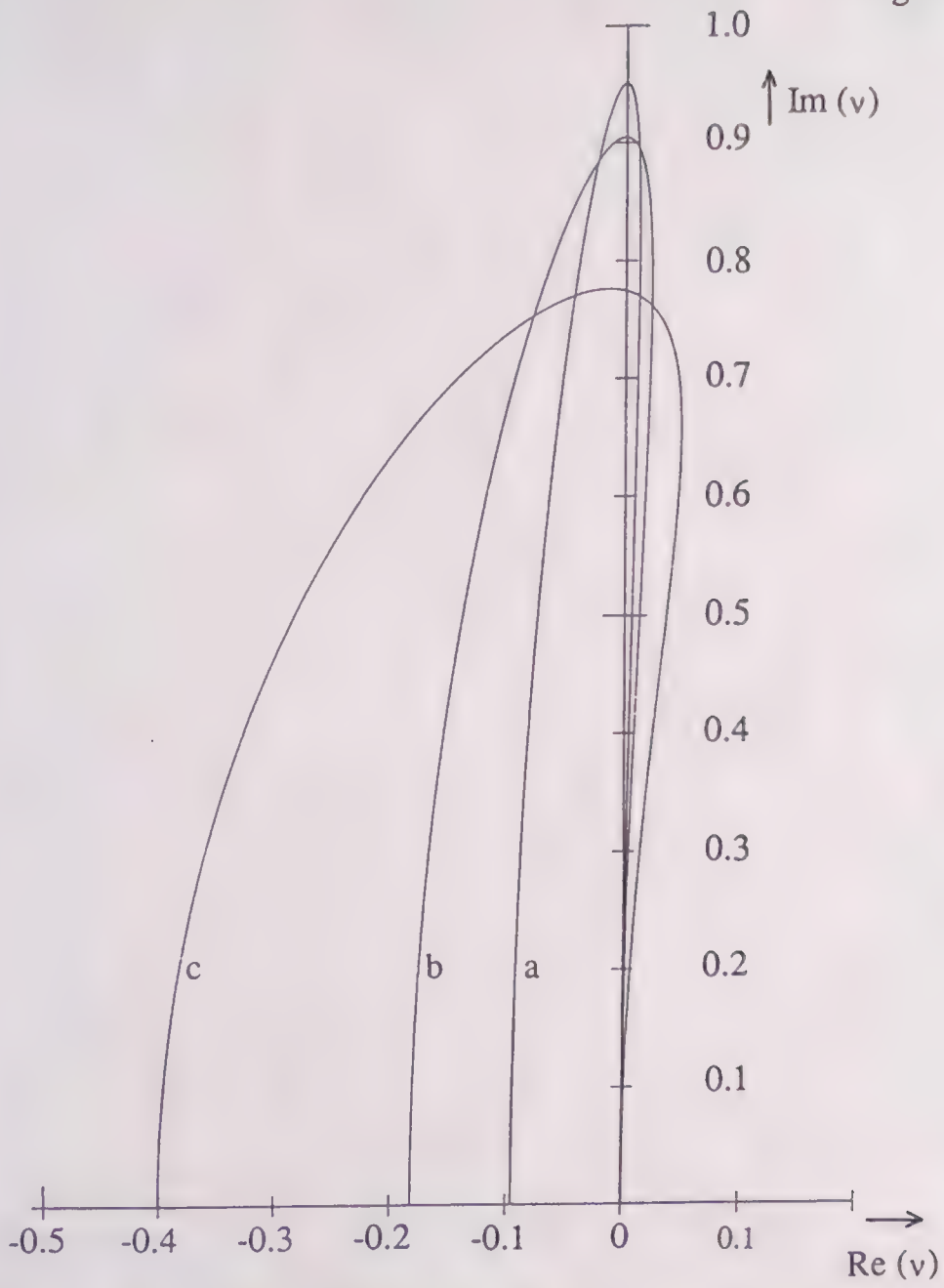


Figure 3a

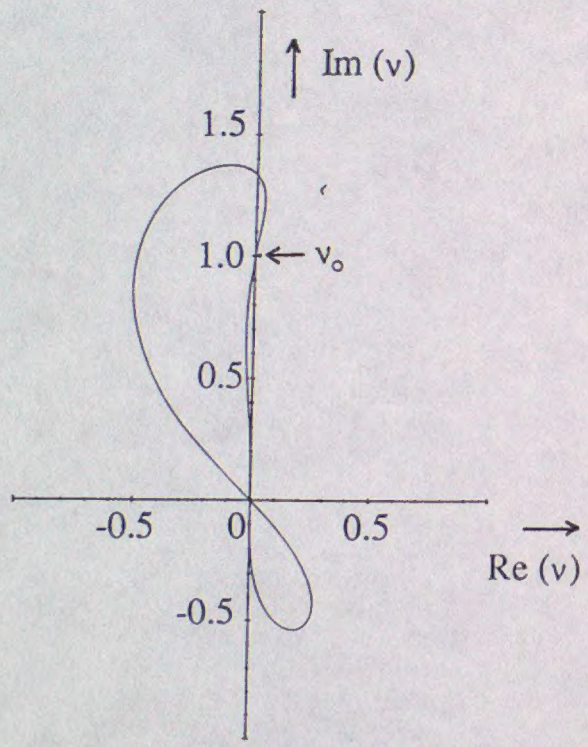


Figure 3b

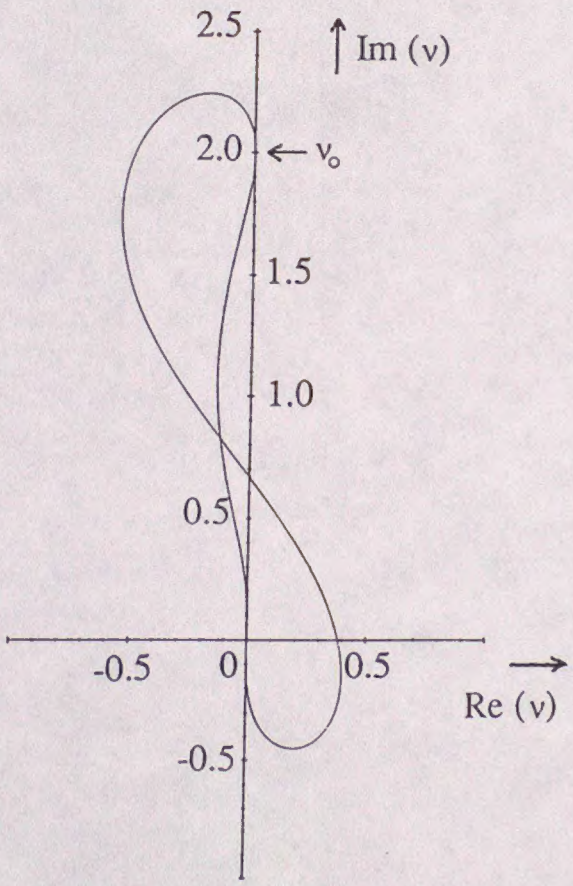


Figure 3c

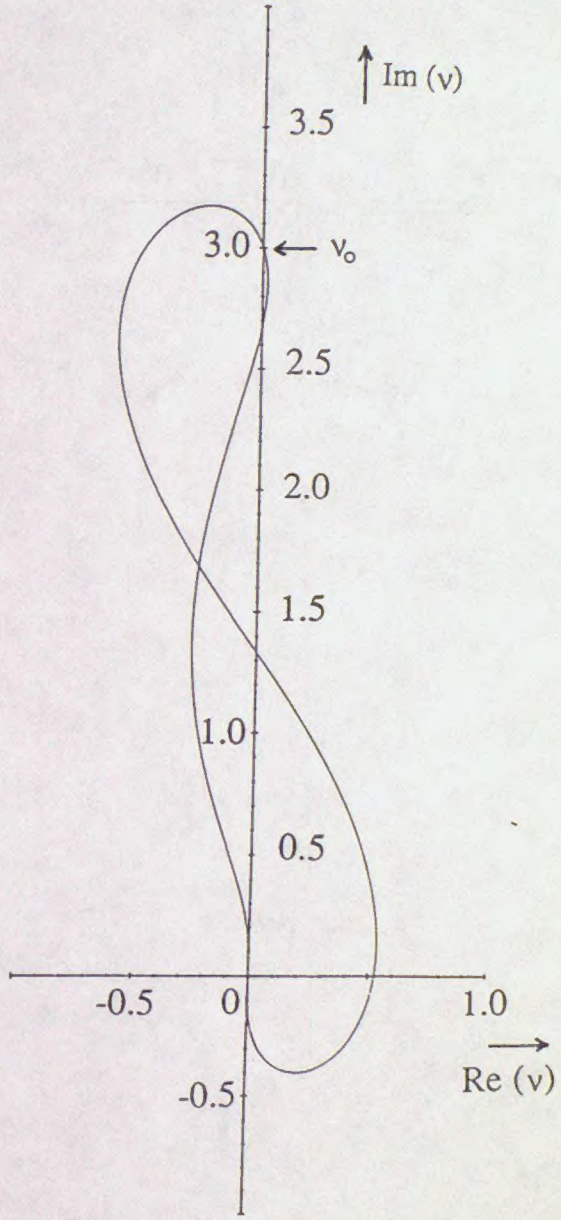


Figure 4a

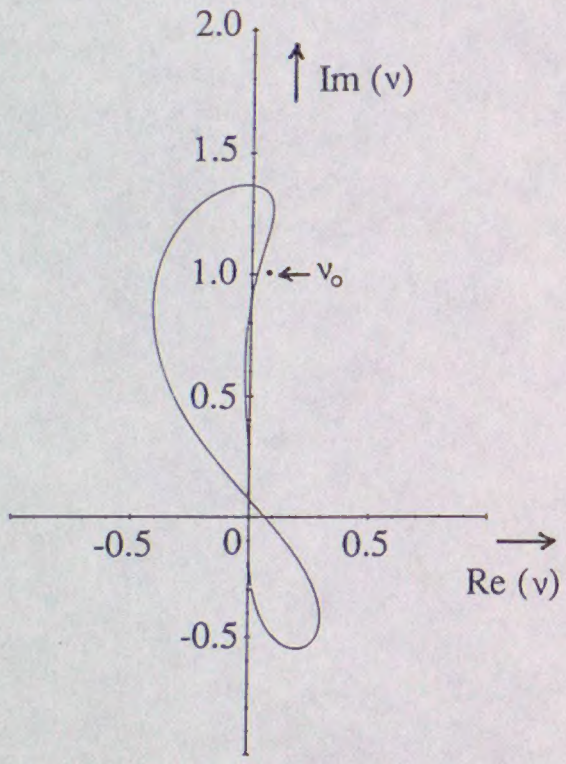


Figure 4b

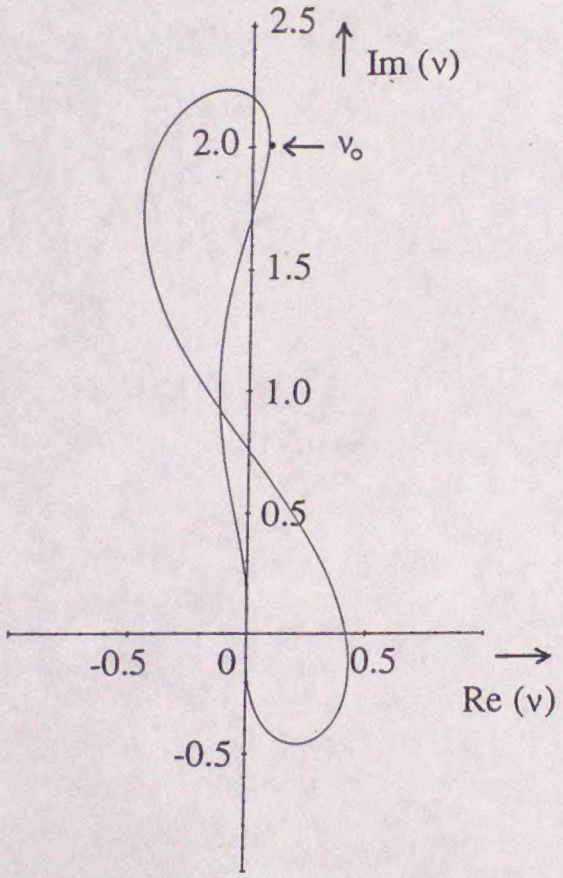
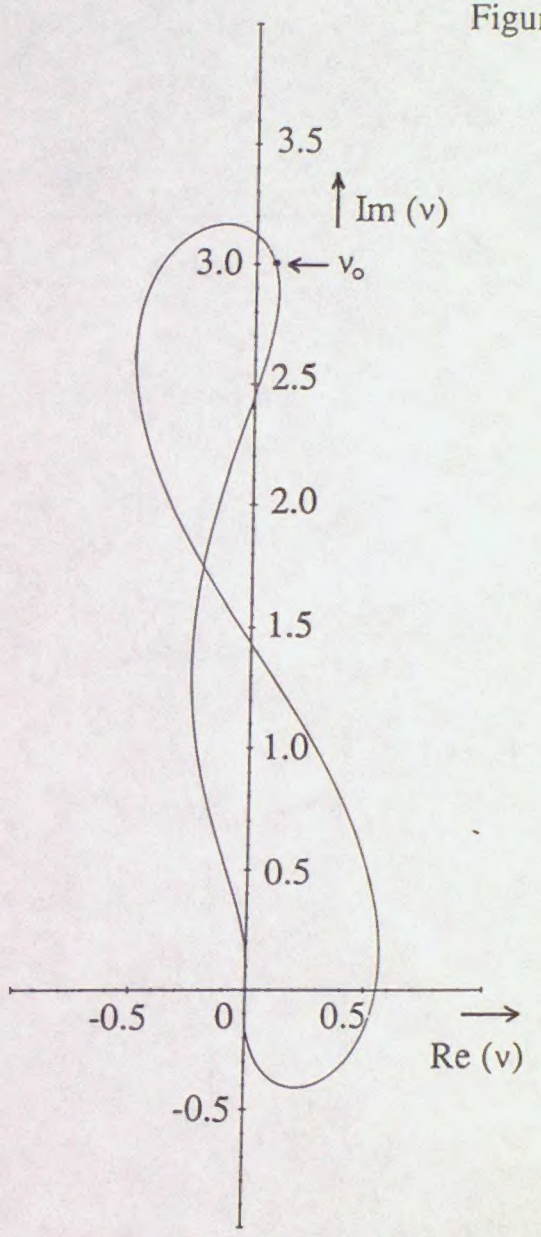


Figure 4c



CURRENT SHORT - RANGE FORECASTING RESEARCH SCIENTIFIC NOTES (AUGUST 1990)

The Short- Range Forecasting Research Branch (formerly Met O 11) Scientific Notes which contain information of current use are listed below. The complete set of Scientific Notes is available from the National Meteorological Library on loan, if required.

1. **The theory of periodic solutions of the semi-geostrophic equations.**
R.J. Purser
October 1987

2. **Properties of the partial differential equations governing various types of atmospheric motions and implications for numerical methods.**
M.J.P. Cullen
December 1987

3. **A geometric model of balanced, axisymmetric flows with embedded penetrative convection.**
G.J. Shutts, M. Booth and J. Norbury
February 1988

4. **Implicit finite difference methods for computing discontinuous atmospheric flows.**
M.J.P. Cullen
March 1988

5. **Variational aspects of semi-geostrophic theory.**
R.J. Purser
August 1988

6. **On the incorporation of atmospheric boundary layer effects into a balanced model.**
M.J.P. Cullen
July 1988

7 **Implicit finite difference methods for modelling discontinuous atmospheric flows.**
M.J.P. Cullen
June 1988

8. **An analytical model of the growth of a frontal intrusion.**
M.W. Holt and G.J. Shutts
November 1988

9. **Planetary semi-geostrophic equations derived from Hamilton's principle.**
G.J. Shutts
July 1988

10. **Semi-geostrophic moist frontogenesis in a Lagrangian model.**
M.W. Holt
September 1988

11. **Generalised Lagrangian solutions for atmospheric and ocean flows.**
M.J.P. Cullen, J. Norbury and R.J. Purser
November 1988

12. **Properties of the Lagrangian semi-geostrophic equations.**
M.J.P. Cullen and R.J. Purser
January 1989

13. **A simple two phase precipitation scheme for use in numerical weather prediction models.**
B.W. Golding
June 1989

14. **A test of a semi-implicit integration technique for a fully compressible non-hydrostatic model.**
M.J.P. Cullen
September 1989

15. **Dynamical aspects of the October storm 1987 : A study of a successful Fine-mesh simulation**
G.J. Shutts
September 1989

16. **A conservative split-explicit integration scheme suitable for forecast and climate models.**
M.J.P. Cullen and T. Davis
January 1990

17. **The dynamical structure of some North Atlantic depressions simulated by the fine-mesh model.**
G.J.Shutts
January 1990

18. **An Idealised Simulation of the Indian Monsoon using Primitive Equation and Quasi - Equilibrium models.**
M.H. Mawson and M.J.P. Cullen
August 1990

19. **Time - Integration Schemes for meteorological simulations involving oscillatory and dissipative modes.**
R.J. Purser
July 1990